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Periodic economic cycles: the effect of evolution towards criticality, and control

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Abstract—When a system becomes unstable or noise becomes excessive, often regulations of the form of limiters (barriers obstructing an excursion into undesired directions) are imposed. It is hoped that by the influence of this element, the system can be calmed and its behavior optimized. We consider a simple noisy nonlinear economics model that self-organizes towards criticality. We demonstrate that the inherent effect of limiters is the emergence of stable cycles, and that the limiters need to be implemented with care in order to obtain an optimized system response. In particular, implementing the limiter at maximal system response is generally a suboptimal solution. We find that the system average is generally optimized by controlling a period-one cycle. Furthermore, we provide optimality conditions for the case that the control is restricted to be on the natural system behavior.

1. Introduction

Economic booms and bouts affect modern societies thoroughly, with a direct impact on the individual's biographies. In western economies, cycles have been an ubiquitous (undesired) observation. Among the most remarkable, the Kitchin cycles emerged [1]. Until the 1970s, as the legacy of Keynes [2], cycles were regarded as primarily due to variations in demand (company investments and household consumptions). As a consequence, economic analysis focused on monetary and fiscal measures to offset demand shocks. During the 1970s, it became obvious that stabilization policies based on this theory failed. Shocks on the supply side, in the form of rising oil prices and declining productivity growth, emerged to be equally crucial for the generation of cycles. In a paper published in 1982, Kydland and Prescott [3] offered new approaches to the control of macroeconomic developments. One of their conclusions was that the control should be kept constant throughout a cycle, in order to minimize negative effects.

Cycles and crises may be inherent to the principles on which our economics are based. However, if they could be predicted and their origin understood, they might be engineered to take a softer course. An extreme form of this approach was taken in the centrally planned economies in the former socialist countries. In order to deal with this problem in democratic societies, it is necessary to be able to communicate a sufficiently simple optimality policy. For obtaining it, the understanding of the response to control in simple economic models may provide important guidelines [4]. The prediction problem of economics is closely related to the one in chaotic processes, where strategies to overcome it have been developed. Although the question to which extent real economies can be classified as chaotic can readily be disputed, low-dimensional chaotic models yield insight into the mechanisms that govern the response of economics to control policies. Interestingly, already in early implementations of the optimal control program, it was found that control mechanisms themselves may induce chaotic behavior [5] and render optimal control impossible. As a general mechanism inherent in many of these examples, chaos is induced by a preference function that depends on past experience. This delay mechanism naturally makes a dynamical system infinite-dimensional, which has the tendency of resulting in a chaotic behavior.

In our contribution, we identify a general principle that naturally generates cycles in economical models. We then demonstrate a detailed mechanism of how cycles are additionally introduced when applying even the simplest control strategies. The obtained insights add a new facet to the control advice by Kydland and Prescott [3]: The optimal system behavior is not obtained by controlling on the natural cycle, but is achieved by a controlled period-one orbit. This not only requires a control policy that is kept fixed through time. To acquire the period-one state, a strong initial control effort is generally required, and control must permanently be maintained.

2. A simple view on economics

When exponential growth is possible, real economies have little problems. It is mostly when the limits of the economic systems are reached that their prediction becomes difficult. From the mathematical point of view, this is due to the nonlinearities that are required to keep the system within the boundaries. Economies naturally tend towards the recruitment of all available resources. This drives them towards the boundaries and fosters a natural tendency of the system to evolve towards maximally developed nonlinearities. We thus can describe economics in a simplified and abstract way in terms of a parameter indicating the degree of globalization of resources (nonlinearity parameter a), and a dynamical parameter x expressing a generalized consumption. The evolution of this simple model of economics takes place on three time scales: a slow one which modifies parameter a, an intermediate-term variable x that is assumed to be deterministic, and momentary perturbations that are included in x in the form of noise. The underlying deterministic system is defined by the property that for states far from full exploitation of the resources, the consumption can grow almost linearly. Close to maximal exploitation, the next consumption is required to be small, to let the system recover.

A most simple and generic setting to model this dynamics is provided by the iterated logistic map,

$$f: \quad [0,1] \to [0,1]: \quad x_{n+1} = a x_n (1 - x_n). \quad (1)$$

The above-mentioned self-organization towards an ever growing exploitation of the phase-space $[0,1]$, is reflected in a slow increase of the order parameter a towards $a = 4$. If a is increased further, large-scale erratic behavior sets in, as the process is no longer confined to the previously invariant unit interval. After a potentially chaotic transient, the system settles in a new area of stability, where the same scenario takes place anew, starting at rescaled small a. We believe that in particular the effects by technical shocks may adequately be described in this framework. On its way towards the globalization of resources ($a \to 4$), the system undergoes a continued period-doubling bifurcation route, where a stable period-one solution is transformed, over a cascade of stable orbits of increasing orders 2^n (where $n = 2, 3, 4 \ldots$), into a chaotic solution (Feigenbaum period-doubling cascade [6]). Using renormalization theory, it can be shown that in order to reach the next bifurcation, a progresses geometrically, with factor $q \approx 1/4.67$. The properties exhibited by the logistic map are characteristic for a large universality class of unimodal maps. Our model is thus characteristic for the whole class of systems that are subject to such a process of self-organization.

3. Effects of simple control

Whereas the usage of the logistic map as a simple, yet generic, model of macroscopic economics seems reasonably motivated, in real economics the demand x is characterized by strong short-term fluctuations, often of local or external origin. Whereas in the case of small a such perturbations are stabilized by the system itself, for larger a they lead to ever more long-lived erratic excursions. To incorporate these fluctuations within our model, we perturb x by multiplicative noise, for simplicity chosen uniformly distributed over a finite interval. The size of the noise sampling interval, in the following denoted by *str*, is a measure for the amount of noise. To render economics predictable under these circumstances, it is natural to apply a control mechanism to x. For this, a sufficiently simple control tool is needed, whose properties are well understood and which does not additionally complicate the behavior of the system. As a natural candidate, control by means of a limiting value on x that the system is not allowed to cross, can be – and in reality often is – imposed.

Recently, exact results for this so-called hard limiter control (HLC) have been obtained. For reasons of convenience, we will expose their nontrivial essence. By introducing a limiter, orbits that sojourn into the forbidden area are eliminated (see Fig. 1). Modified in this way, the system tends to replace previously chaotic with periodic behavior. By gradually restricting the phase-space, it is possible to transfer initially chaotic into ever simpler periodic motion. When the modified system is tuned in such a way that the control mechanism is only marginally effective, the controlled orbit runs in the close neighborhood of an orbit of the uncontrolled system. In a series of papers [7, 8, 9], this control approach was successfully applied in different experimental settings, and its properties were fully analyzed.

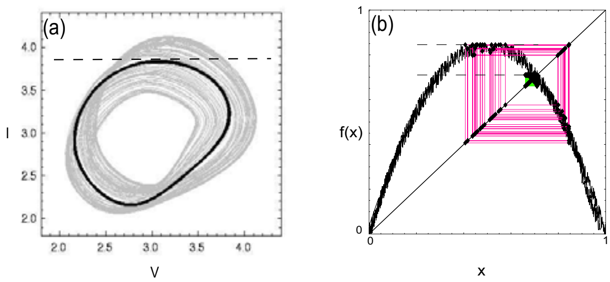

Figure 1: HLC for time-continuous and discrete dynamical systems. Limiter positions are indicated by dashed lines. a) HLC changes chaotic into period-one behavior (modified from [7]). b) HLC for the noisy logistic map. Placement of the limiter around the maximum of the map preserves the natural noisy period-two orbit (red). For lower placement, a modified period-one behavior is obtained (green).

Flat-topped maps are the proper paradigm for studying HLC [8, 9]. They are obtained by replacing the peak region of a map by a horizontal line at height h, which limits the phase-space to $\{x \mid x < h\}$. A detailed analysis shows that the class of flat-topped maps shares a number of remarkable topological and metric features [8]. It is observed that as a function of the control strength, the controlled map undergoes a period-doubling bifurcation cascade, leading to long, seemingly chaotic, orbits. However, in this system, no chaotic orbits are allowed. By ergodicity, each orbit will eventually pass by the control segment, from where on the orbit is periodic, as landing on the control segment entrains a zero slope

Period-doubling cascades are characterized by two constants, α and δ [6]. The constant α describes the asymptotic scaling of the fork openings of subsequent period doublings, whereas δ represents the scaling of the intervals of period 2^n to that of period 2^{n-1} near the period-doubling accumulation point, i.e. at the transition to chaos. The observed period-doubling bifurcation cascades are typical for flat-topped maps (or, the control method) and differ in scaling from the Feigenbaum case. The ratio of the bifurcation fork openings within forks of the same periodicity now depends on the derivative of the map, and is therefore non-universal.

In applications, the time required to arrive in a close neighborhood of the target orbit is an important characteristic of the control method. With the classical methods, unstable periodic orbits can only be controlled when the system is already in the vicinity of the target orbit. As the initial transients can become very long, algorithms have been designed to speed up this process [10]. HLC renders targeting algorithms obsolete, as the control-time problem is equivalent to a strange repeller-escape (control is achieved, as soon as the orbit lands on the flat top). As a consequence, the convergence onto the selected orbit is exponential [8].

These properties of 1-d HLC systems fully describe the effects generated by the limiter control. Due to the control, only periodic behavior is possible. Period doubling cascades that have a superexponential scaling $\delta^{-1}(n) \sim 2^{-2^n}$ [8] and therefore are not of the Feigenbaum type, emerge in the control space. The convergence onto controlled orbits is exponential. Controlled orbits are unmodified original orbits only at bifurcation points of the controlled map. For generic one-parameter families of maps, all bifurcation points are regular, isolated in a compact space, and as a consequence, have zero Lebesgue measure. These properties substantially modify the uncontrolled system behavior.

4. Natural vs. control-induced cycles; control results

It is a wide-spread misunderstanding that control methods only apply to inherently unstable systems. Unmodified control methods can be used to control on unstable orbits of inherently stable systems. In either case, the control should be only minimally active. In the noise-free case, the control is optimal, if after an initial phase the controller does no longer experience any noticeable strain. This is the case at the bifurcation points of the controlled map. Questions that remain are whether a corresponding statement also holds true for noisy systems, and on which of the orbits should be controlled.

As the economic system evolves, it will be in a noisy, but stable period-one state. This is a convenient economic behavior. Predictions and forecasts are simple to make. To reduce the noise, the limiter will be placed around the periodic point. As the system turns into a period-two, the question emerges whether to maintain the unstable period-one cycle, or whether to move on to the stable period-two. We argue that maintaining the period-one cycle is preferable, from most economics aspects. The predictions of these systems are simpler, and lead to simpler economic policies. Many economic indicators (taxes, budgets, etc.) are evaluated over a period of one year. Moreover, the period-one x-average will be generally higher than that of the controlled period-two, as well as of any other higher-order cycle. This appears counter-intuitive, since the natural tendency to relax to the "natural" system state has to be compensated for by the control. From the convexity of the nonlinear map, however, it is easy to prove that our claim holds. To change a natural higher-order periodic behavior into a period-one state generally requires a relatively strong initial control action. That this is beneficial appears to be counter-intuitive again, and needs to be communicated in an accompanied economic policy statement. When the time scale over which the external parameter a varies becomes comparable to the cycle's wavelength, the optimality of the aforedescribed control may break down, as continued adjustments need to be made in order to follow the changing location of the period-one. In this case, it may be preferential to control on a natural cycle. The most obvious control goal would then be to control the system as closely as possible along the underlying noise-free system. In the numerical control results presented below, we deal with both control goals. To measure the efficacy in performing control on natural cycles, we define the control distance as the absolute difference between the "natural" underlying solution and the controlled solution, per step.

Control in the stable system regime: For our numerical investigations, we restrict ourselves to the control on superstable orbits (by choosing $a = 2$ and $a = 1 + 5^{1/2}$, for the periods one and two, respectively), and apply the control at the cycle maximum. As a measure of efficacy, we calculate the average deviation of the noisy controlled relative to the noise-free system, denoted by *dev*, as a function of the noise and of the limiter position h. This seems to reflect best the natural tendency of the system to return into the vicinity of the uncontrolled noise-free system once the control is relaxed. We find that for zero noise, $dev(h)$ is a piecewise linear function (shown in Fig. 2a for the period-one orbit), where the nonzero slope, associated with h below the maximum of the function f, is determined by the periodicity and by the amount of nonlinearity expressed by a. For nonzero noise, the formerly piecewise linear func-

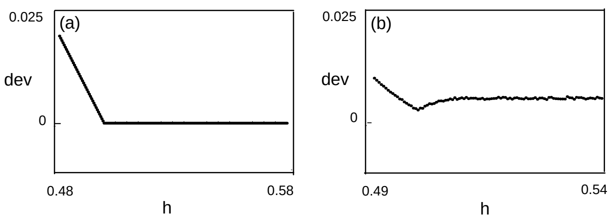

Figure 2: Dependence of *dev* on the control point h (summation over 500 orbit points, period-one orbit). a) For zero noise, a piecewise linear function with a minimum (= optimal noise-free control point) emerges. b) In the presence of noise ($str = 0.02$), the function becomes nonlinear, with a nonzero minimum at the optimal noise-free control point.

tion becomes nonlinear, with the minimum being situated at the optimal control point of the noise-free system (see Fig. 2b). For noise strengths $str < 0.1$, which we consider to be a realistic case, the deviation is a linear function of str. The control on the superstable period-two orbit yields a similar picture. Control is lost, when due to the noise, interchange of orbit points occurs. This is the reason why in the presence of a substantial amount of noise, only low-order cycles can be controlled. For a period-four orbit, already a noise level of $str > 0.01$ leads to control loss. Interestingly, the function $dev(h, str)$ scales linearly with str (identical curves emerge, if h and dev are replaced by h/str and dev/str, respectively). As a rule of thumb, by means of optimal control, the deviation can be reduced by a factor of ~ 0.5.

Control in the chaotic system regime: To investigate the control in the chaotic regime, we focus on the fully developed logistic map ($a = 4$). To control on true system orbits, the control point must be chosen at locations corresponding to the bifurcation points, whose location can be evaluated analytically [8]. Without control, chaos prevents the system from staying on a given cycle. As a consequence, the efficacy of the control is measured as the difference between controlled noise-free and controlled noisy systems. In order to obtain a period-one orbit in the noise-free case, the limiter was adjusted to $h = 0.75$. Experiments show that in the presence of noise, the optimal control point moves away from the noise-free optimal control point. This is in contrast to the behavior in the stable regime, and may help to distinguish between the two cases. The displacement δh is a linear function of the noise strength, as is the deviation dev measured at the optimal shifted control point. For period 2, the optimal control point's displacement and the minimal deviation are again linear in the noise strength (see Fig. 3a,b). The shift of the control point extends over an interval of more than $\delta h = 0.01$, and therefore is of a size comparable to the added noise.

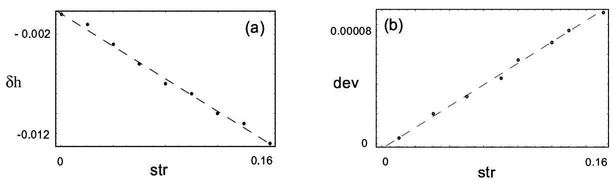

Figure 3: Results for the chaotic regime, where an unstable period-two orbit is controlled. a) Linear dependence of the optimal control point displacement δh on the noise strength str. b) Linear dependence of dev at the optimal control point, on the noise strength str.

5. Conclusions

Control mechanisms of limiter type are common in economics. This control, however, inherently generates super-stable system behavior, whether the underlying behavior be periodic or chaotic. *A priori*, a frequent change of the position of the limiter might appear to be a suitable strategy in order to compensate for the amplified or newly created cyclic behavior. This strategy, however, will only result in ever more erratic system behavior. Our analysis shows that it is advantageous to keep the limiter fixed, adjusting it only over time-scales where the system parameter a changes noticeably. In this way, reliable cycles of smaller periodicity will emerge. Among these cycles, period-one appears to be the optimal one, from most economic points of view. To recruit this state, a strong initial intervention is necessary and the control must be permanent. In discussions of real economics, these properties will be natural arguments against the proposed control. To overcome such arguments, a sufficiently simple control policy must be formulated in democratic societies. The discussed framework may provide the basis for the formulation of a control policy to attain economic optimality.

We emphasize that short-term cycles emerge on all levels of economics. It has become, e.g., a common observation, that the demands for certain professionals (in central Europe in particular for teachers) undergo large fluctuations, from one year to another. In one year, severe problems are encountered in recruiting a sufficient number, so that the professional requirements have to be lowered, whereas in the next year, there is an excess supply. We propose to interpret this as the signature of an economy that has moved out of period-one behavior. From the teaching quality as well as from the individual's biographies points of view, the occurrence of this effect should be prevented or smoothened. Our approach offers a perspective for understanding, studying and, potentially also engineering, such phenomena.

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